PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 28, Fasc. 1 (2008), pp. 21–40

# CONVERGENCE IN VARIATION OF THE JOINT LAWS OF MULTIPLE STABLE STOCHASTIC INTEGRALS

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*Abstract.* In this note, we are interested in the regularity in the sense of total variation of the joint laws of multiple stable stochastic integrals. Namely, we show that the convergence

$$\mathcal{L}\big(I_{d_1}(f_1^n), \ldots, I_{d_p}(f_p^n)\big) \xrightarrow{\text{var}} \mathcal{L}\big(I_{d_1}(f_1), \ldots, I_{d_p}(f_p)\big) \quad \text{as } n \to +\infty$$

holds true as long as each kernel $f_i^n$ converges when $n \to +\infty$ to $f_i$ in the Lorentz-type space $L^\alpha(\log_+)^{d_i-1}([0,1]^{d_i})$ for $1 \leqslant i \leqslant p$. This result generalizes [4] from the one-dimensional case to the joint law case. It generalizes also [6] from the Wiener–Itô setting to the stable setting and [5] in the study of joint law of multiple stable integrals.

**2000 AMS Mathematics Subject Classification:** Primary: 60F99, 60G52, 60H05; Secondary: 60G57.

**Key words and phrases:** Convergence in variation, multiple stochastic integrals, stable process, LePage representation, method of superstructure.

## 1. INTRODUCTION

In this paper, we deal with the regularity of the joint laws of multiple stable integrals (MSIs)

$$I_d(f) = \int_{[0,1]^d} f\,dM^d \tag{1.1}$$

with respect to their integrand $f$. Here and in the sequel, $M$ is an $\alpha$-stable random measure on $\big([0,1], \mathcal{B}([0,1])\big)$ defined for $0 < \alpha < 2$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$:

$$M(A) \stackrel{\mathcal{L}}{=} S_\alpha\left(\lambda(A)^{1/\alpha}, \frac{\int_A \beta d\lambda}{\lambda(A)}, 0\right), \quad A \in \mathcal{B}([0,1]),$$

where $\lambda$ is the Lebesgue measure and $\beta : [0,1] \to [-1,1]$ is the skewness intensity of $M$ (see Samorodnitsky and Taqqu [14], Section 3). Moreover, for $\alpha \geqslant 1$, the measure $M$ is assumed to be symmetric (that is, $\beta = 0$).

The MSIs are a generalization of multiple Wiener–Itô integrals (MWIs). A broad literature is devoted to the study of MWIs and it is natural to investigate which properties of MWIs remain true in the stable case.

The MSI in (1.1) is defined for kernel $f$ in a Lorentz-type space:

$$f \in L^\alpha(\log_+)^{d-1}([0,1]^d) := \Big\{ f : [0,1]^d \to \mathbb{R} \,\Big|\, \int_{[0,1]^d} |f|^\alpha (1 + \log_+ |f|)^{d-1} \, d\lambda^d \Big\},$$

where $\log_+ x := \log(x \vee 1)$. The main feature of MSI is given by the representation theorem which gives an insight into the discrete structure of MSI. It shows that $I_d(f)$ can be represented in law by a multiple LePage-type series

$$S_d(f) = C_\alpha^{d/\alpha} \sum_{i_1, \ldots, i_d > 0} \gamma_{i_1} \ldots \gamma_{i_d} \Gamma_{i_1}^{-1/\alpha} \ldots \Gamma_{i_d}^{-1/\alpha} f(V_{i_1}, \ldots, V_{i_d}), \tag{1.2}$$

where $C_\alpha = \big( \int_0^\infty x^{-\alpha} \sin x \, dx \big)^{-1}$ is a normalization factor, $(\Gamma_i)_{i>0}$ is the sequence of arrival times of a standard Poisson process and $(V_i, \gamma_i)_{i>0}$ are independent and identically distributed random vectors with $V_i$ uniformly distributed on $[0,1]$ and $\gamma_i = \pm 1$ with conditional laws

$$\mathbb{P}(\gamma_i = -1 \mid V_i) = \frac{1 - \beta(V_i)}{2}, \quad \mathbb{P}(\gamma_i = +1 \mid V_i) = \frac{1 + \beta(V_i)}{2}.$$

Moreover, the sequences $(\Gamma_i)_{i>0}$ and $(V_i, \gamma_i)_{i>0}$ are independent.

The representation theorem shows that MSIs are also related to random multilinear forms (see [10]). For a complete account on the construction of MSI, we refer to [3] and references therein. The laws of MSIs have been studied by several authors. We briefly review some results on the law of MSIs.

In [13], the tail of $I_d(f)$ is expressed in terms of $f$.

In [12], the regularity of the sample path of a process defined by an integral like in (1.1) is related to the smoothness of the kernel.

In [11], the independence of MSIs is studied in terms of the kernels, generalizing the MWI case of [15].

In [5], the existence of the densities for the joint laws of MSIs is studied, generalizing the MWI case of [7].

In this article, we go further in the study of the joint laws of MSIs than in [5] and we study their regularity in the sense of total variation norm. More precisely, given $d_1, \ldots, d_p$, the dimensions of $p$ MSIs, we study the convergence in variation of the joint laws

$$\big( I_{d_1}(f_1^n), \ldots, I_{d_p}(f_p^n) \big) \tag{1.3}$$

for integrands

$$f_1^n \to f_1 \text{ in } L^\alpha(\log_+)^{d_1-1}([0,1]^{d_1}), \ldots, \; f_p^n \to f_p \text{ in } L^\alpha(\log_+)^{d_p-1}([0,1]^{d_p}). \tag{1.4}$$

This is a generalization of [4] which deals with one-dimensional law of MSIs ($p = 1$ in our setting). This is also a generalization of [6] where the convergence in variation of joint laws is investigated for MWIs ($\alpha = 2$ in our setting). In this paper, we deal with arbitrary $p \in \mathbb{N}^*$ and arbitrary $\alpha \in (0, 2)$.

Moreover, since the densities of the joint laws of MSIs exist (under rather broad conditions, see [5]), the convergence in variation of the law states also the convergence of the densities in $L^1(\mathbb{R}^d)$.

The paper is organized as follows. In Section 2, we start giving some notation already used in [5]; they will be applied all along this article. Next, we state the convergence result in Theorem 2.1. The sequel is devoted to the proof of Theorem 2.1. The problem is first reduced in Sections 3 and 4. In Section 5, we use the method of superstructure to reduce to the study of finite-dimensional functionals. Finally, in Section 6, the convergence in variation of these functionals is shown using the results of convergence in variation for smooth image-measures from [1] (see Proposition 2.1).

Note that the one-dimensional argument used in [4] (which states the one-dimensional counterpart of Theorem 2.1) cannot be generalized in a multidimensional setting (at least easily). Actually, the proof of Theorem 2.1 relies on arguments already used in [6] and [5]. But this is not a simple rewriting of these arguments. Indeed, they have to be merged together: on the one hand, the method of stratification used in [5] is not sufficient to yield a convergence in variation, instead we use the method of superstructure, on the other hand, the argument in [6] relies on Gaussian analysis which has to be replaced by stable considerations. Moreover, new difficulties appear in the implementation of these merged arguments.

In the sequel a.s. stands for *almost surely*, a.e. for *almost everywhere*, i.i.d. for *independent and identically distributed*, $:=$ means a definition, $C$ is a finite and positive generic constant, $\mu_A$ is the restriction to a measurable set $A$ of a measure $\mu$, $\|\nu\|$ is the total variation of a signed measure $\nu$, $\xrightarrow{\text{var}}$ stands for the convergence of variation, $\xrightarrow{P}$ for the convergence in probability $P$, and bold characters are used for multi-indicial notation.

## 2. CONVERGENCE IN VARIATION OF JOINT LAWS

In this study, we shall use the same background as in [5]. We begin by recalling the notation of [5] that will be used all along this article:

for $i = 1, \ldots, p$, $N_i = d_1 + \ldots + d_i$, $N = N_p$;
$\mathbf{a}^i = (a_0^i, \ldots, a_p^i) \in \mathbb{N}^{p+1}$ a $(p+1)$-partition of $d_i = |\mathbf{a}^i| = a_0^i + \ldots + a_p^i$;
$\mathbf{a} = (\mathbf{a}^1, \ldots, \mathbf{a}^p) \in (\mathbb{N}^{p+1})^p$;
$M_{\mathbf{a}} = (a_j^i)_{1 \leqslant i,j \leqslant p}$ a $p$-square matrix with

$$d_i = \sum_{k=0}^{p} a_k^i \ \text{ for } 1 \leqslant i \leqslant p \quad \text{and} \quad b_k = \sum_{i=1}^{p} a_k^i \ \text{ for } 0 \leqslant k \leqslant p;$$

for $\mathbf{b} = (b_1, \dots, b_p) \in \mathbb{N}^p$:

$$E(\mathbf{b}) = \big\{\mathbf{a} = (\mathbf{a}^1, \dots, \mathbf{a}^p) \;\big|\; a_1^i + \dots + a_p^i = d_i,\ \sum_{i=1}^{p} a_k^i = b_k,\ \ k = 1, \dots, p\big\};$$

$\sigma_{\mathbf{a}}$ the permutation of $\{1, \dots, N\}$ that sends for each $i, k$

$$j = \sum_{u=1}^{k-1} b_u + \sum_{s=1}^{i-1} a_k^s + l, \qquad l = 1, \dots, a_k^i,$$

to

$$\sigma_{\mathbf{a}}(j) = \sum_{v=1}^{i-1} d_v + \sum_{s=1}^{k-1} a_s^i + l;$$

$U_{\mathbf{a}} : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ associated with $\sigma_{\mathbf{a}}$ by

$$U_{\mathbf{a}}(t_1, \dots, t_N) = (t_{\sigma_{\mathbf{a}}(1)}, \dots, t_{\sigma_{\mathbf{a}}(N)});$$

$\Pi_{b_1,\dots,b_d}$ the subgroup of $\Pi_N$ consisting of permutations preserving the following "$\mathbf{b}$-blocks": $(1, \dots, b_1)$, $(b_1 + 1, \dots, b_1 + b_2)$, $\dots$, $(b_1 + b_2 + \dots + b_{p-1} + 1, \dots, b_1 + b_2 + \dots + b_p = N)$:

$$S_{b_1,\dots,b_d}\phi_{\mathbf{b}}(t) = \frac{b_1! \dots b_d!}{N!} \sum_{\sigma \in \Pi_{b_1,\dots,b_d}} \sum_{\mathbf{a} \in E(\mathbf{b})} \prod_{i=1}^{p} \frac{d_i!}{a_0^i! \dots a_p^i!}\ \det M_{\mathbf{a}}\ \phi\big(U_{\mathbf{a}}(t)\big),$$

where $\phi(t) = \phi(t_1, \dots, t_N) = f_1(t_1, \dots, t_{N_1}) \dots f_p(t_{N_{p-1}+1}, \dots, t_{N_p})$; note that the function $S_{b_1,\dots,b_d}\phi_{\mathbf{b}}$ is symmetric in each $\mathbf{b}$-block.

The main result of this paper is:

THEOREM 2.1. *Let $f_1^n, \dots, f_p^n$ be kernels converging to $f_1, \dots, f_p$, respectively, as in* (1.4). *Suppose moreover that the limit functions $f_1, \dots, f_p$ satisfy the following hypothesis*:

$$(\mathbf{H}) \begin{cases} S_{b_1,\dots,b_d}\phi_{\mathbf{b}} \neq 0 \textit{ a.e. on } [0,1]^N \\ \textit{for some } \mathbf{b} = (b_1, \dots, b_p) \in (\mathbb{N}^*)^p \textit{ with } |\mathbf{b}| = N = d_1 + \dots + d_p. \end{cases}$$

*Then* $\mathcal{L}\big(I_{d_1}(f_1^n), \dots, I_{d_p}(f_p^n)\big) \xrightarrow{var} \mathcal{L}\big(I_{d_1}(f_1), \dots, I_{d_p}(f_p)\big)$ *when* $n \to +\infty$.

Roughly speaking, (**H**) is a non-degeneracy condition dealing with how the $f_i$'s are overlapped. The same remark and the same examples as in [5] apply to condition (**H**). In particular, we stress on that this condition is optimal in several examples and coincides with the condition for the same convergence for joint laws of MWIs; see [6]. For instance:

for $p = 1$ and $d_1 = d$, (**H**) is satisfied with $\mathbf{b} = d$ if $f \neq 0$ a.e.;

for $p > 0$ and $d_1 = \dots = d_p = 1$, (**H**) is satisfied with $\mathbf{b} = (1, 1)$ if

$$\det \big\{ \big(f_i(t_j)\big)_{1 \leqslant i,j \leqslant p} \big\} \neq 0 \ \text{ a.e.};$$

for $p = 2$ and $d_1 = d_2 = 2$, $(\mathbf{H})$ is satisfied with $\mathbf{b} = (2, 2)$ if $f_1$ and $f_2$ are not proportional a.e.

The global scheme of the proof is the following. First, we explain how to reduce the problem in Section 3 (see (3.5)). Using the representation of stable integrals by LePage series, we introduce on the Skorokhod space some related functionals to be studied (see (3.2)). Next, approximating and localizing the problem in Section 4, we use the method of superstructure in Section 5 where the main point is to study the convergence in variation of measures under finite-dimensional mappings (see (5.9)). This is finally done in Section 6 with the following result from [1] and the study of some related coefficients (see (6.2)).

PROPOSITION 2.1 (Corollary 4 in [1]). *Let* $F_j, F \in W^{p,1}_{loc}(\mathbb{R}^n, \mathbb{R}^n)$, *where* $p \geqslant n$, *and let the mappings* $F_j$ *converge to* $F$ *with respect to the Sobolev norm* $\|\cdot\|_{p,1}$ *on every ball. Assume that* $E \subset \{\det DF \neq 0\}$ *is a set of finite Lebesgue measure. Then* $\lambda_{|E} F_j^{-1} \xrightarrow{var} \lambda_{|E} F^{-1}$.

### 3. REDUCTION OF THE PROBLEM

In this section, we describe the arguments used in [3] and [5] to reduce the study of the convergence in variation of laws as in (1.3).

**Representation and stable stuff.** Like in [5], we first reduce the study to random multiple LePage series. From the representation theorem ([3], Theorem 3.2), we have as in (1.2) the following equality of joint laws:

$$\big(S_{d_1}(f_1), \dots, S_{d_p}(f_p)\big) \stackrel{\mathcal{L}}{=} \big(I_{d_1}(f_1), \dots, I_{d_p}(f_p)\big)\,. \tag{3.1}$$

Moreover, we have from [3], Sections 4.1.2 and 4.2.3, the following:

PROPOSITION 3.1. *Let* $\big(I_{d_1}(f_1^n), \dots, I_{d_p}(f_p^n)\big)$ *be a vector of MSIs with kernels* $f_1^n, \dots, f_p^n$ *converging as in* (1.4). *Then, when* $n \to +\infty$, *we have*

$$\big(I_{d_1}(f_1^n), \dots, I_{d_p}(f_p^n)\big) \xrightarrow{\mathbb{P}} \big(I_{d_1}(f_1), \dots, I_{d_p}(f_p)\big).$$

By (3.1), we shall actually study the joint law of $\big(S_{d_1}(f_1^n), \dots, S_{d_p}(f_p^n)\big)$. For $x$ in the Skorokhod space $\mathbb{D}$ (the space of càdlàg functions on $[0, 1]$), let $\delta_x(t)$ be the jump of $x$ at $t$ and let $(t_i)_{i>0}$ be the list of its jump times. We consider the multidimensional functional $F = (F_1, \dots, F_p)$ with $F_i : \mathbb{D} \longrightarrow \mathbb{R}$ given by

$$F_i(x) = \sum_{t_1,\dots,t_{d_i}} \delta_x(t_1) \dots \delta_x(t_{d_i})\, f_i(t_1, \dots, t_{d_i}) \tag{3.2}$$

whenever the multiple series is convergent; otherwise $F_i(x) = 0$.

In the sequel, we shall also consider the stable standard process $\eta$ given by

$$\eta_t = M([0, t]), \quad t \in [0, 1].$$

The sample paths of $\eta$ live in $\mathbb{D}$; we denote its law by $P$. By the representation theorem (in a one-dimensional case) we have

$$\eta_t = \int_{[0,1]} \mathbf{1}_{[0,t]}\, dM \overset{\mathcal{L}}{=} C_\alpha^{1/\alpha} \sum_{i>0} \gamma_i {\Gamma_i}^{-1/\alpha}\, \mathbf{1}_{[0,t]}(V_i)$$

from which the following interpretations come:

$V_i$, $i > 0$, are the jump times of the stable process $\eta$;

$C_\alpha^{1/\alpha} \Gamma_i^{-1/\alpha}$ is the modulus of the jump at $V_i$, decreasingly ordered;

$\gamma_i$ indicates the direction of the jump.

We deduce

$$\begin{aligned} F_i\big(\eta_{\bullet}(\omega)\big) &= C_\alpha^{d_i/\alpha} \sum_{k_1,\ldots,k_{d_i}>0} (\gamma_{k_1}\Gamma_{k_1}^{-1/\alpha}) \ldots (\gamma_{k_{d_i}}\Gamma_{k_{d_i}}^{-1/\alpha})\, f_i(V_{k_1},\ldots,V_{k_{d_i}}) \\ &= S_{d_i}(f_i)(\omega), \end{aligned}$$

so that

$$F(\eta) \overset{\mathcal{L}}{=} \big(S_{d_1}(f_1),\ldots,S_{d_p}(f_p)\big).$$

We define also $F^n$ from $(f_1^n,\ldots,f_p^n)$ like $F$ from $(f_1,\ldots,f_p)$ in (3.2). The convergence in variation of the law of (1.3) actually rewrites, in our notation:

$$P(F^n)^{-1} \xrightarrow{\text{var}} PF^{-1} \quad \text{as } n \to +\infty. \tag{3.3}$$

In the sequel, we shall use the following result which is Proposition 3.1 written in terms of the functionals related to the corresponding MSIs.

PROPOSITION 3.2. *Let $f_1^n,\ldots,f_p^n$ be converging kernels as in* (1.4). *Then, with the previous notation, we have*

$$F^n \xrightarrow{P} F \quad \textit{as } n \to +\infty.$$

**Approximation.** This procedure consists in the following straightforward result:

PROPOSITION 3.3 (Approximation). *In order to prove* (3.3), *it is enough to see that for all $\varepsilon > 0$ there is some measurable set $\mathbb{D}(\varepsilon)$ in $\mathbb{D}$ with $P\big(\mathbb{D}(\varepsilon)\big) > 1-\varepsilon$ and*

$$P_{\mathbb{D}(\varepsilon)}(F^n)^{-1} \xrightarrow{var} P_{\mathbb{D}(\varepsilon)}F^{-1}. \tag{3.4}$$

Proof. We have

$$\begin{aligned} &\|P(F^n)^{-1} - PF^{-1}\| \\ &\qquad \leqslant \|P(F^n)^{-1} - P_{\mathbb{D}(\varepsilon)}(F^n)^{-1}\| + \|P_{\mathbb{D}(\varepsilon)}(F^n)^{-1} - P_{\mathbb{D}(\varepsilon)}F^{-1}\| \\ &\qquad\quad + \|P_{\mathbb{D}(\varepsilon)}F^{-1} - PF^{-1}\| \\ &\qquad \leqslant 2P\big(\mathbb{D}(\varepsilon)^c\big) + \|P_{\mathbb{D}(\varepsilon)}(F^n)^{-1} - P_{\mathbb{D}(\varepsilon)}F^{-1}\| \\ &\qquad \leqslant 2\varepsilon + \|P_{\mathbb{D}(\varepsilon)}(F^n)^{-1} - P_{\mathbb{D}(\varepsilon)}F^{-1}\|. \end{aligned}$$

But by (3.4) the last expression is bounded by $3\varepsilon$ for $n$ large enough. This completes the proof of the argument of approximation. ■

**Localization.** Using the separability of $\mathbb{D}(\varepsilon)$, we localize the problem by the following result:

PROPOSITION 3.4 (Localization). *In order to prove* (3.4), *it is enough to exhibit for all $x \in \mathbb{D}(\varepsilon)$ some neighbourhood $V(x)$ of $x$ such that*

$$P_{V(x)}(F^n)^{-1} \xrightarrow{var} P_{V(x)}F^{-1} \quad \text{as } n \to +\infty. \tag{3.5}$$

Proof. Since $\mathbb{D}(\varepsilon)$ is separable, there is a countable family $\{x_i, i \in \mathbb{N}^*\}$ such that $\mathbb{D}(\varepsilon) = \bigcup_{i=1}^{+\infty} V(x_i)$. We have $\lim_{k\to+\infty} P\big(\bigcup_{i=1}^{k} V(x_i)\big) = P\big(\mathbb{D}(\varepsilon)\big)$ so that for any fixed $\epsilon > 0$ and $k$ large enough it follows that $P\big(\mathbb{D}(\varepsilon) \setminus A_k\big) < \epsilon$, where $A_k = \bigcup_{i=1}^{k} V(x_i)$. Therefore, for such a $k$, we have

$$\begin{aligned}
&\|P_{\mathbb{D}(\varepsilon)}(F^n)^{-1} - P_{\mathbb{D}(\varepsilon)}F^{-1}\| \\
&\qquad \leqslant \|P_{\mathbb{D}(\varepsilon)}(F^n)^{-1} - P_{A_k}(F^n)^{-1}\| + \|P_{A_k}(F^n)^{-1} - P_{A_k}F^{-1}\| \\
&\qquad\quad + \|P_{A_k}(F)^{-1} - P_{\mathbb{D}(\varepsilon)}F^{-1}\| \\
&\qquad \leqslant 2P\big(\mathbb{D}(\varepsilon) \setminus A_k\big) + \|P_{A_k}(F^n)^{-1} - P_{A_k}F^{-1}\| \\
&\qquad \leqslant 2\epsilon + \sum_{i=1}^{k} \|P_{V(x_i)}(F^n)^{-1} - P_{V(x_i)}F^{-1}\| \leqslant 3\epsilon,
\end{aligned}$$

where the last bound comes for $n$ large enough from (3.5). Finally, we derive (3.4), and so the localization is proved. ■

Using localization, it is enough to prove (3.5). To do so, we shall use the method of superstructure. For a general description of this method, we refer to [9]. Note that, like in [3] and [5] with the method of stratification, these preliminary procedures of approximation and localization are necessary in order to implement successfully the method of superstructure.

## 4. APPROXIMATION AND LOCALIZATION

In this section, we exhibit the set $\mathbb{D}(\varepsilon)$ required in the approximation procedure and the neighbourhood $V(x)$ required for $P$-almost all $x \in \mathbb{D}(\varepsilon)$ in the localization procedure. Actually, the approximation and localization procedures are the same as in [5], we thus refer to Section 3 of [5] for a precise description. Here, we only sketch the main steps.

**Approximation.** Let $\mathbf{b}$ be given by hypothesis (**H**) in Theorem 2.1 and $\tilde{t} = (\tilde{t}_1, \ldots, \tilde{t}_N)$ be some Lebesgue point of the set

$$A_{\mathbf{b}} = \{t \in [0,1]^N \mid S_{b_1,\ldots,b_p}\phi_b(t) \neq 0\} \in \mathcal{B}([0,1]^N).$$

The Lebesgue measure of $A_{\mathbf{b}}$ is positive by hypothesis (**H**). There is no restriction in assuming that $\tilde{t}$ is chosen with its coordinates all distinct ($\tilde{t}_i \neq \tilde{t}_j$, $i \neq j$). Let $\varepsilon > 0$ be fixed; there is a product neighbourhood $V_\varepsilon = U_1^\varepsilon \times \ldots \times U_N^\varepsilon$ of $\tilde{t}$ in $[0,1]^N$ satisfying

$$U_i^\varepsilon \cap U_j^\varepsilon = \emptyset,\ i \neq j, \quad \text{and} \quad \frac{\lambda^N(V_\varepsilon \cap A_{\mathbf{b}})}{\lambda^N(V_\varepsilon)} \geqslant 1 - \varepsilon. \tag{4.1}$$

We consider the following sets:

$$\begin{aligned}\tilde{\mathbb{D}}(\varepsilon) = \{x \in \mathbb{D} \mid &\text{ for } i = 1, 2, \ldots, N,\ x \text{ has at least one jump at a time in } U_i^\varepsilon,\\ &\text{the maximal modulus of these jumps being realized only once}\},\\ \mathbb{D}(\varepsilon) = \big\{x \in \tilde{\mathbb{D}}(\varepsilon) \mid &\ x \text{ has a unique maximal jump on each } U_i^\varepsilon \text{ at } T_{U_i^\varepsilon}(x)\\ &\text{with } T_\varepsilon(x) := \big(T_{U_1^\varepsilon}(x), \ldots, T_{U_N^\varepsilon}(x)\big) \in A_{\mathbf{b}}\big\}.\end{aligned}$$

We recall from [5] the following result for the standard stable process $\eta$:

LEMMA 4.1. *The random vector $T_\varepsilon(\eta) = \big(T_{U_1^\varepsilon}(\eta), \ldots, T_{U_N^\varepsilon}(\eta)\big)$ is uniformly distributed on $V_\varepsilon$. Moreover, for $i \neq j$, $T_{U_i^\varepsilon}(\eta)$ and $T_{U_j^\varepsilon}(\eta)$ are independent.*

With (4.1), Lemma 4.1 gives:

$$P\big(\mathbb{D}(\varepsilon)\big) = P_{\tilde{\mathbb{D}}(\varepsilon)} T_\varepsilon^{-1}(A_{\mathbf{b}}) = \frac{\lambda^N(V_\varepsilon \cap A_{\mathbf{b}})}{\lambda^N(V_\varepsilon)} \geqslant 1 - \varepsilon.$$

The set $\mathbb{D}(\varepsilon)$ is the set required in the procedure of approximation of $\mathbb{D}$.

**Localization.** For the sake of completeness of the notation, we recall the localization procedure of [5]. Let $x \in \mathbb{D}(\varepsilon)$ be fixed and put for $i = 1, \ldots, N$:
$t_i = T_{U_i^\varepsilon}(x)$ the time of the largest jump of $x$ in $U_i^\varepsilon$;
$t_i'$ the time of the second largest jump of $x$ in $U_i^\varepsilon$, $|\delta_x(t_i')| < |\delta_x(t_i)|$;
$\varepsilon_0 = \frac{1}{2}\min_{i=1,\ldots,N} |\delta_x(t_i)|$.

Note that, by Lemma 4.1, the jump time $t_i$ can be seen as a random variable on $\Big(\mathbb{D}(\varepsilon), P_{\mathbb{D}(\varepsilon)}/P\big(\mathbb{D}(\varepsilon)\big)\Big)$ whose law is uniform on $U_i^\varepsilon$.

By finiteness of the number of jumps of $x$ larger than $\varepsilon_0/2$, we select $\delta_1 > 0$ such that $t_i$ is the unique time of $\Delta_i' := (t_i - \delta_1, t_i + \delta_1) \subset U_i^\varepsilon$ where a jump larger than $\varepsilon_0/2$ in modulus occurs. Let the following technical conditions be fulfilled:

$$\varepsilon_0/2 < \varepsilon_1 < \varepsilon_2 < \ldots < \varepsilon_p < \varepsilon_0; \tag{4.2}$$

$$\delta_2 < \frac{1}{4}\min\big\{\varepsilon_0, 2\delta_1, \inf_{i=1,\ldots,N}\{|\delta_x(t_i)| - |\delta_x(t_i')|\}, 2\varepsilon_1 - \varepsilon_0\big\}; \tag{4.3}$$

$$\beta = \delta_1 - \delta_2 \quad (\delta_2 \leqslant \beta \leqslant \delta_1); \quad \Delta_i := (t_i - \beta, t_i + \beta) \subset \Delta_i' \subset U_i^\varepsilon.$$

In the sequel, we consider a local field $l = (l_x)_{x\in\mathbb{D}}$. That is, for $m \in \mathbb{N}^*$, $\varepsilon > 0$, reals $\tau_i$, and intervals $\Delta_i = (a_i, b_i)$, we define

$$l_x(t) = \sum_{s:\ \delta_x(s)>\varepsilon} \Big(\sum_{i=1}^m \tau_i \mathbf{1}_{\Delta_i}(s)\mathbf{1}_{[s,\infty[}(t)\Big)^+ - \sum_{s:\ \delta_x(s)<-\varepsilon} \Big(\sum_{i=1}^m \tau_i \mathbf{1}_{\Delta_i}(s)\mathbf{1}_{[s,\infty[}(t)\Big)^-;$$

see [9], p. 163, for a precise definition of local fields. Roughly speaking, local fields $(l_x)_x$ are admissible directions for stable processes. Moreover, we define

$$\omega_x(t) = \begin{cases} \tau_i & \text{if } t \in (a_i, b_i),\ |\delta_x(t)| > \varepsilon,\ \delta_x(t)\,\tau_i > 0, \\ 0 & \text{otherwise,} \end{cases} \tag{4.4}$$

so that the jumps of $x$ and $x + c\,l_x$ are linked by $\delta_{x+c\,l_x}(t) = \delta_x(t) + c\,\omega_x(t)$.

Next, we associate with $l$ the following sets:

$A(l)^+$ is the set of $x \in \mathbb{D}$ such that, for all $i$ with $\tau_i > 0$, $x$ does not have jumps of length exactly $\varepsilon$ on $(a_i, b_i)$, $\delta_x(a_i) < \varepsilon$, $\delta_x(b_i) < \varepsilon$, and $x$ has at least one jump larger than $\varepsilon$ on $(a_i, b_i)$;

$A(l)^-$ is the set of $x \in \mathbb{D}$ such that, for all $i$ with $\tau_i < 0$, $x$ does not have jumps of length exactly $-\varepsilon$ on $(a_i, b_i)$, $\delta_x(a_i) > -\varepsilon$, $\delta_x(b_i) > -\varepsilon$, and $x$ has at least one jump lower than $-\varepsilon$ on $(a_i, b_i)$;

the set $A(l)$ is given by

$$A(l) = A(l)^+ \cap A(l)^-. \tag{4.5}$$

The set $A(l)$ is suitable to study the local field $l$. In particular, it is shown in [5], Sections A1 and A2, that $A(l)$ is open in $\mathbb{D}$ and that the local field $l$ is continuous on $A(l)$.

To apply the method of superstructure in a multidimensional setting, we consider $p$ local fields $l^i$, $1 \leqslant i \leqslant p$, and their corresponding open set $A(l^i)$, given as in (4.5). We select the $p$ local fields with the following parameters: for $i = 1, \ldots, p$,

$\varepsilon_i$ given by (4.2);

$m_i = b_i,$ given by hypothesis (**H**);

$\Delta^i_j = \Delta_{b_1+\ldots+b_{i-1}+j}$ for $j = 1, \ldots, b_i$;

$\tau^i_j$ with the same sign as $\delta_x(t_i)$ and with constant modulus $\tau > 0$.

In the sequel, we put $\ell = (l^1, \ldots, l^p)$. We have $x \in \tilde{A}(\ell) := \bigcap_{i=1}^p A(l^i)$, an open set. We shall apply the localization procedure with the following neighbourhood $V(x)$:

$$V(x) = B(x, \delta_2) \cap \tilde{A}(\ell) \cap \mathbb{D}(\varepsilon), \tag{4.6}$$

where $\delta_2$ is given in (4.3).

Finally, with $\mathbb{D}(\varepsilon)$ and $V(x)$ given above, Propositions 3.3 and 3.4 apply and the proof of Theorem 2.1 reduces to showing (3.5). The convergence (3.5) is tackled with the method of superstructure in the next sections.

## 5. SUPERSTRUCTURE IN $\mathbb{D}(\varepsilon)$

In order to prove (3.5), we use the method of superstructure in the neighbourhood $V(x)$ of $x$ defined by (4.6). For a general account on this method we refer to [9], Section 5. Here, we only sketch the method.

This method applies to study the convergence $PF_n^{-1} \xrightarrow{\text{var}} PF^{-1}$ when $F_n$ and $F$ are some functionals on some $(\mathcal{Y}, P)$. When we have a family of transformations $(G_{\mathbf{c}})_{\mathbf{c}\in(\mathbb{R}_+)^p}$ satisfying

$$PG_{\mathbf{c}}^{-1} \xrightarrow{\text{var}} P, \quad \mathbf{c} \to 0, \tag{5.1}$$

we define the following auxiliary measures and functionals on the product space $\mathcal{Y}_\epsilon = [0,\epsilon]^p \times \mathcal{Y}$, $\epsilon > 0$:

$$Q_\epsilon = \frac{1}{\epsilon^p}\lambda_{[0,\epsilon]^p} \otimes P, \quad F_\epsilon(\mathbf{c}, y) = F\big(G_{\mathbf{c}}(y)\big)$$

(note that $F_\epsilon$ depends on $\epsilon$ only through its domain of definition $\mathcal{Y}_\epsilon$).

Since

$$Q_\epsilon F_\epsilon^{-1} = \frac{1}{\epsilon^p} \int_{[0,\epsilon]^p} PG_{\mathbf{c}}^{-1}F^{-1} d\mathbf{c}$$

for the total variation, we derive

$$\|Q_\epsilon F_\epsilon^{-1} - PF^{-1}\| \leqslant \frac{1}{\epsilon^p} \int_{[0,\epsilon]^p} \|P - PG_{\mathbf{c}}^{-1}\|\, d\mathbf{c},$$

and from (5.1) together with the dominated convergence we get

$$\lim_{\epsilon\to 0} \|Q_\epsilon F_\epsilon^{-1} - PF^{-1}\| = 0. \tag{5.2}$$

Next, we express $Q_\epsilon F_\epsilon^{-1}$ as a mixture of finite-dimensional measures: we put $\varphi_y(\mathbf{c}) = F\big(G_{\mathbf{c}}(y)\big)$ for $\mathbf{c} \in [0,\epsilon]^p$ and we have

$$Q_\epsilon F_\epsilon^{-1} = \frac{1}{\epsilon^p} \int_{\mathcal{Y}} \lambda_{[0,\epsilon]^p} \varphi_y^{-1} dP. \tag{5.3}$$

In the sequel, we shall note $\varphi_y = (\varphi_y^1, \dots, \varphi_y^p)$. In our setting, in order to study the laws of $\big(I_{d_1}(f_1^n), \dots, I_{d_p}(f_p^n)\big)_n$ for $f_i^n \to f$ when $n \to +\infty$, in the space $L^\alpha(\log_+)^{d_i-1}([0,1]^{d_i})$, $1 \leqslant i \leqslant p$, we apply this method to $\mathcal{Y} = \tilde{A}(\ell)$ equipped with the restricted probability law $P_x := P_{V(x)}$ of the process $\eta$ and to the functionals $F$ and $F^n$ given in (3.3). We use the family of transformations $(G_{\mathbf{c}})_{\mathbf{c}\in(\mathbb{R}_+)^p}$ defined from the local fields $l^i$, $i = 1, \dots, p$, by

$$G_{\mathbf{c}} : \begin{cases} \tilde{A}(\ell) \longrightarrow \tilde{A}(\ell), \\ \quad y \longmapsto y + \langle \mathbf{c}, l_y \rangle, \end{cases} \tag{5.4}$$

where $\langle \mathbf{c}, l_y \rangle := c_1\, l_y^1 + \ldots + c_p\, l_y^p$ for $\mathbf{c} = (c_1, \ldots, c_p)$. Moreover, with a similar notation, observe that with $\omega^i$ defined from $l^i$ as in (4.4) we have for $\mathbf{c} \in (\mathbb{R}_+)^p$

$$\delta_{G_{\mathbf{c}}(y)}(t) = \delta_y(t) + \langle \mathbf{c}, \omega_y \rangle. \tag{5.5}$$

REMARK 5.1. The open set $\tilde{A}(\ell)$ is invariant under $G_{c_1,\ldots,c_p}$. Roughly speaking, this is because each term $l_y^i$, $1 \leqslant i \leqslant p$, emphasizes the membership of $A(l^i)$ and does not alter conditions to belong to $A(l^j)$ for $j \neq i$. A more detailed justification is given in [3].

Our condition (5.1) is satisfied for the family of transformations (5.4). Indeed, in Lemma 4.1 of [3] it is shown that $(G_{\mathbf{c}})_{\mathbf{c}}$ defined an admissible semigroup in the sense of [9], p. 14. Moreover, the conditional measures of $P$ on the orbits of the semigroup $(G_{\mathbf{c}})_{\mathbf{c}}$ have densities. Since $(G_{\mathbf{c}})_{\mathbf{c}}$ acts as a translation, we derive (5.1) when $\mathbf{c} \to 0$ (see (21.8) in [9] for a more detailed proof). The convergence in (5.1) is enough for our study but we have actually more:

LEMMA 5.1. *The convergence $P^x G_{\mathbf{c}}^{-1} \xrightarrow{var} P^x$ is uniform with respect to $x$ in $\tilde{A}(\ell)$.*

Proof. In this proof only, we use the setting described in Section 4 of [9]. First, we define an equivalence relation $\mathcal{R}$ on $\tilde{A}(\ell)$ by $x_1 \mathcal{R} x_2$ if and only if there are $\mathbf{c}^1, \mathbf{c}^2 \in (\mathbb{R}_+)^p$ such that $G_{\mathbf{c}^1} x_1 = G_{\mathbf{c}^2} x_2$. Let $\Gamma$ be the partition given by $\mathcal{R}$ and $\pi : \tilde{A}(\ell) \to \tilde{A}(\ell)/\Gamma$ be the canonical projection. The equivalence classes $\pi^{-1}(\gamma), \gamma \in \tilde{A}(\ell)/\Gamma$, are called *orbits* of the semigroup $(G_{\mathbf{c}})_{\mathbf{c}}$. In Proposition 4.2 of [9], each orbit $\pi^{-1}(\gamma)$ is shown to be isomorphic (via a mapping $J_\gamma$) to a measurable set $C_\gamma \subset \mathbb{R}^p$. Next, we define a Lebesgue measure $\lambda_\gamma$ on $\pi^{-1}(\gamma)$ by

$$\lambda_\gamma(B) = \lambda_p\Big(J_\gamma\big(B \cap \pi^{-1}(\gamma)\big)\Big), \quad B \in \mathcal{B}\big(\tilde{A}(\ell)\big),$$

where $\lambda_p$ is the Lebesgue measure on $\mathbb{R}^p$. Moreover, the mapping $J_\gamma$ intertwines the action of the semigroup $(G_{\mathbf{c}})_{\mathbf{c}}$ on the orbit $\pi^{-1}(\gamma)$ with the action of the semigroup of translation $(\tau_{\mathbf{c}})_{\mathbf{c}}$ on $C_\gamma$:

$$J_\gamma G_{\mathbf{c}} = \tau_{\mathbf{c}} J_\gamma.$$

Thus, we have

$$\begin{aligned}
\|P^x G_{\mathbf{c}}^{-1} - P^x\| &= \Big\| \int_{\tilde{A}(\ell)/\Gamma} P_\gamma^x G_{\mathbf{c}}^{-1}\, dP_\Gamma(\gamma) - \int_{\tilde{A}(\ell)/\Gamma} P_\gamma^x\, dP_\Gamma(\gamma) \Big\| \\
&\leqslant \int_{\tilde{A}(\ell)/\Gamma} \|P_\gamma^x G_{\mathbf{c}}^{-1} - P_\gamma^x\|\, dP_\Gamma(\gamma) \\
&\leqslant \int_{\tilde{A}(\ell)/\Gamma} \|P_\gamma^x J_\gamma^{-1} \tau_{\mathbf{c}}^{-1} J_\gamma - P_\gamma^x J_\gamma^{-1} J_\gamma\|\, dP_\Gamma(\gamma) \\
&\leqslant \int_{\tilde{A}(\ell)/\Gamma} \|P_\gamma^x J_\gamma^{-1} \tau_{\mathbf{c}}^{-1} - P_\gamma^x J_\gamma^{-1}\|\, dP_\Gamma(\gamma).
\end{aligned} \tag{5.6}$$

In Theorem 4.1 of [9], it is shown that the conditional measures $(P_\gamma)$ of $P$ on the orbits of the semigroup $(G_{\mathbf{c}})_{\mathbf{c}}$ have densities. Thus $P_\gamma J_\gamma^{-1}$ has also a density in $C_\gamma \subset \mathbb{R}^p$. But the translation operator is uniformly continuous in $L^1(\mathbb{R}^p)$ (that is, $\lim_{h\to 0}\varphi(\cdot + h) = \varphi(\cdot)$ in $L^1(\mathbb{R}^p)$ uniformly with respect to $\varphi$).

Therefore,

$$\lim_{\mathbf{c}\to 0} \|P_\gamma^x J_\gamma^{-1}\tau_{\mathbf{c}}^{-1} - P_\gamma^x J_\gamma^{-1}\| = 0$$

holds true uniformly with respect to both $\gamma$ and $x$. Integrating with respect to $\gamma \in \tilde{A}(\ell)/\Gamma$, we derive that the right-hand side of (5.6) goes to 0 when $\mathbf{c} \to 0$ uniformly with respect to $x$. Finally, (5.1) holds uniformly with respect to $x$ in $\tilde{A}(\ell)$. ∎

Applying the method of superstructure in this setting, we define as previously multidimensional auxiliary functionals $F_\epsilon^n$ on $\mathcal{Y}_\epsilon$ and we derive as in (5.2):

$$(5.7)\quad \|Q_\epsilon(F_\epsilon^n)^{-1} - P^x(F^n)^{-1}\| \leqslant \frac{1}{\epsilon^p}\int\limits_{[0,\epsilon]^p} \|P^x - P^x G_{\mathbf{c}}^{-1}\|\, d\mathbf{c} \to 0 \quad \text{as } \epsilon \to 0$$

uniformly with respect to $n \in \mathbb{N}$. We have

$$\begin{aligned}(5.8)\quad &\|P^x F^{-1} - P^x(F^n)^{-1}\| \\ &\leqslant \|P^x F^{-1} - Q_\epsilon F_\epsilon^{-1}\| + \|Q_\epsilon F_\epsilon^{-1} - Q_\epsilon(F_\epsilon^n)^{-1}\| + \|Q_\epsilon(F_\epsilon^n)^{-1} - P^x(F^n)^{-1}\|.\end{aligned}$$

We deduce from (5.2) and (5.7) that the first and third terms in (5.8) can be chosen arbitrarily small for $\epsilon > 0$ small enough and uniformly with respect $n$. Note that even if we will not use this, by Lemma 5.1, it holds also uniformly with respect to $x$. Consequently, it remains to deal, when $\epsilon > 0$ is fixed, with the second term on the right-hand side of (5.8) when $n \to +\infty$. Moreover, from (5.3) and its counterpart for index $n$, we can write

$$\|Q_\epsilon F_\epsilon^{-1} - Q_\epsilon(F_\epsilon^n)^{-1}\| \leqslant \frac{1}{\epsilon^p}\int\limits_{V(x)} \|\lambda_{[0,\epsilon]^p}\varphi_y^{-1} - \lambda_{[0,\epsilon]^p}\varphi_{n,y}^{-1}\|\, dP$$

with $\varphi_{n,y}(\mathbf{c}) = \big(\varphi_{n,y}^1(\mathbf{c}), \dots, \varphi_{n,y}^p(\mathbf{c})\big) = F^n\big(G_{\mathbf{c}}(y)\big)$. Note that the domain of integration above is $V(x)$ (and not $\tilde{A}(\ell)$) because the method of superstructure is applied on $\tilde{A}(\ell)$ with $P_x = P_{V(x)}$. It is enough now to show for $P$-almost all $y \in V(x)$ that

$$(5.9)\qquad \lambda_{[0,\epsilon]^p}\varphi_{n,y}^{-1} \xrightarrow{\text{var}} \lambda_{[0,\epsilon]^p}\varphi_y^{-1} \quad \text{as } n \to +\infty.$$

This is done in Section 6 using Proposition 2.1.

## 6. STUDY OF THE CONDITIONAL FUNCTIONALS

After some algebraic calculations, we re-express the functionals $\varphi_{n,y}$ and $\varphi_y$ as ordered polynomials. The conditional functional $\varphi_y : \mathbb{R}^p \longrightarrow \mathbb{R}^p$ is given by

$$\begin{aligned}\varphi_y(\mathbf{c}) &= \big(\varphi_{1,y}(\mathbf{c}), \ldots, \varphi_{p,y}(\mathbf{c})\big)\\ &= F(y + c_1\, l_y^1 + \ldots + c_p\, l_y^p), \quad \mathbf{c} = (c_1, \ldots, c_p).\end{aligned}$$

Moreover, we have

$$\begin{aligned}\varphi_{i,y}(\mathbf{c}) &= \varphi_i(y + \langle \mathbf{c}, l_y\rangle)\\ &= \sum_{s_1,\ldots,s_{d_i}} \Big(\prod_{j=1}^{d_i} \big(\delta_y(s_j) + \langle \mathbf{c}, \omega_y(s_j)\rangle\big)\Big) f_i(s_1, \ldots, s_{d_i}),\end{aligned}$$

where $(s_i)_i$ is the list of the jump times of $y \in \mathbb{D}$. We obtain a polynomial in $c_1, \ldots, c_p$; we can develop it as in Section 4.2 of [5] and, finally, we have

$$\varphi_{i,y}(\mathbf{c}) = \sum_{\substack{\mathbf{a}^i = (a_0^i, \ldots, a_p^i)\\ |\mathbf{a}^i| = d_i}} B(\mathbf{a}^i, y)\, \mathbf{c}^{\mathbf{a}^i}, \tag{6.1}$$

where, in order to simplify the notation, we put

$$\mathbf{c}^{\mathbf{a}^i} = 1^{a_0^i} c_1^{a_1^i} \ldots c_p^{a_p^i} \quad \text{for } \mathbf{a}^i = (a_0^i, a_1^i, \ldots, a_p^i),$$

$$\begin{aligned}B(\mathbf{a}^i, y) = \sum_{\substack{\{I_k\} \text{ partition of}\\ \{1,\ldots,d_i\},\ \mathrm{card}\, I_k = a_k^i}} \sum_{s_1,\ldots,s_{d_i}} \Big(\prod_{j\in I_0} \delta_y(s_j)\Big)\Big(\prod_{j\in I_1} w_y^1(s_j)\Big) \times \ldots\\ \times \Big(\prod_{j\in I_p} w_y^p(s_j)\Big) f_i(s_1, \ldots, s_{d_i})\end{aligned} \tag{6.2}$$

and $(s_j)_j$ is the list of jump times of $y$.

Using the polynomial expression of $\varphi_y^i$, the following key point is shown in Sections 4.2 and 4.3 of [5]. It shall be used later to apply Proposition 2.1 to the measure images $\lambda_{[0,\epsilon]^p} \varphi_{n,y}^{-1}$.

LEMMA 6.1. *Under the hypothesis* $(\mathbf{H})$, *for $P$-almost all $y \in V(x)$, the Jacobian*

$$J_y(\mathbf{c}) := \det\left(\frac{\partial \varphi_y^i}{\partial c_j}(\mathbf{c})\right)_{1\leqslant i,j\leqslant p}$$

*is non-zero for almost all* $\mathbf{c} \in (\mathbb{R}_+)^p$.

Since all functionals $\varphi_{n,y}$ can be developed in the same way, we introduce also the coefficients $B(\mathbf{a}^i, y, n)$ defined as in (6.2) with $f_i^n$ in place of $f_i$, and we study the convergence of $B(\mathbf{a}^i, y, n)$ to $B(\mathbf{a}^i, y)$ when $n \to +\infty$.

In order to simplify the study of $B(\mathbf{a}^i, y, n)$, we begin with the preliminary simpler case of coefficients $B(\mathbf{a}^i, x, n)$ relative to $x \in \mathbb{D}(\varepsilon)$.

**6.1. Study of the coefficients $B(\mathbf{a}^i, x, n)$.**

LEMMA 6.2. *In $\mathbb{D}(\varepsilon)$, we have $B(\mathbf{a}^i, x, n) \xrightarrow{P} B(\mathbf{a}^i, x)$ when $n \to +\infty$.*

Proof. Note that $\prod_{j \in I_1} w_x^1(t_j) \neq 0$ if, for all $j \in I_1$, $t_j \in \Delta_k^1$ for some $1 \leqslant k \leqslant b_1$. Then $w_x^1(t_j) = \pm\tau$ with the same sign as that of the jump $\delta_x(t_j)$. Thus $\prod_{j \in I_1} w_x^1(t_j) = \pm\tau^{a_1^i}$ for $A_{b_1}^{a_1^i} = b_1!/(b_1 - a_1^i)!$ choices of $t_j$, $j \in I_1$. The same holds true for all inner products $\prod_{j \in I_k} w_x^k(t_j) = \pm\tau^{a_k^i}$ for $A_{b_k}^{a_k^i}$ choices of $t_j$, $j \in I_k$, for $1 \leqslant k \leqslant p$. Finally, we have

$$\Big(\prod_{j \in I_1} w_x^1(t_j)\Big) \dots \Big(\prod_{j \in I_p} w_x^p(t_j)\Big) = \pm\tau^{a_1^i + \dots + a_p^i}$$

for $A(\mathbf{a}^i) := A_{b_1}^{a_1^i} \times \dots \times A_{b_p}^{a_p^i}$ choices of index $j$, otherwise the product is zero. Using the symmetry of the kernels $f_i$ and the nullity of $f_i$ on the diagonals, we can thus rewrite

(6.3)
$$B(\mathbf{a}^i, x) = \sum_{\substack{\{I_k\} \text{ partition of} \\ \{1,\dots,d_i\},\ \mathrm{card}\, I_k = a_k^i}}^{(1)} \ \sum_{\substack{A(\mathbf{a}^i) \text{ choices of} \\ t_{a_0^i+1},\dots,t_{d_i}}}^{(2)} \pm\tau^{a_1^i + \dots + a_p^i} \sum_{t_1,\dots,t_{a_0^i}} \Big(\prod_{j \in I_0} \delta_x(t_j)\Big) f_i(t_1, \dots, t_{d_i}).$$

Observe that the outer sums in (6.3), i.e. $\sum^{(1)}$ and $\sum^{(2)}$, are both finite. Moreover, the same computations hold true for the coefficients $B(\mathbf{a}^i, x, n)$ with $f_i^n$ in place of $f_i$.

In order to study the convergence of $B(\mathbf{a}^i, x, n)$ (with respect to $n$), we first deal with the convergence of the inner sum

$$\sum_{t_1,\dots,t_{a_0^i}} \Big(\prod_{j \in I_0} \delta_x(t_j)\Big) f_i^n(t_1, \dots, t_{d_i}) \quad \text{as } n \to +\infty,$$

where $t_{a_0^i+1}, \dots, t_{d_i}$ in $f_i^n$ appear as parameters.

When $a_0^i \neq 0$, this sum can be seen as an MSI like in (3.2). First, since for all $1 \leqslant i \leqslant p$, $f_i^n \to f_i$ in $L^\alpha(\log_+)^{d_i - 1}([0,1]^{d_i})$, taking some subsequence $(n') \subset (n)$, we see that the convergence

$$f_i^{n'}(\cdot, t_{a_0^i+1}, \dots, t_{d_i}) \to f(\cdot, t_{a_0^i+1}, \dots, t_{d_i}) \quad \text{as } n' \to +\infty$$

holds in $L^\alpha(\log_+)^{d_i-1}([0,1]^{a_0^i})$, and thus also in $L^\alpha(\log_+)^{a_0^i-1}([0,1]^{a_0^i})$ for almost all $t_{a_0^i+1}, \dots, t_{d_i}$. Therefore, from Proposition 3.1, when $a_0^i \neq 0$, we have

$$I_{a_0^i}\big(f_i^{n'}(\cdot, t_{a_0^i+1}, \dots, t_{d_i})\big) \xrightarrow{\mathbb{P}} I_{a_0^i}\big(f(\cdot, t_{a_0^i+1}, \dots, t_{d_i})\big) \quad \text{as } n' \to +\infty.$$

Arguing as in Proposition 3.2, we have, when $n' \to +\infty$, for almost all jump times $t_{a_0^i+1}, \dots, t_{d_i}$

(6.4)
$$\sum_{t_1,\dots,t_{a_0^i}} \Big(\prod_{j\in I_0} \delta_x(t_j)\Big) f_i^{n'}(t_1,\dots,t_{d_i}) \xrightarrow{P} \sum_{t_1,\dots,t_{a_0^i}} \Big(\prod_{j\in I_0} \delta_x(t_j)\Big) f_i(t_1,\dots,t_{d_i}),$$

where we recall that $P$ still stands for the law of the stable process $\eta$. Since by Lemma 4.1 the $t_i$'s can be seen as uniform and independent random variables on the $\Delta_i$'s, the following elementary lemma applied with $X = t_1, \dots, t_{a_0^i}$ and $Y = t_{a_0^i+1}, \dots, t_{d_i}$ yields the same convergence in probability as in (6.4) but involving now all the jump times $t_1, \dots, t_{a_0^i}, t_{a_0^i+1}, \dots, t_{d_i}$.

LEMMA 6.3. *Let $X$ and $Y$ be independent random variables and $f_n, f$ be some measurable functions from $\mathbb{R}^2$ to $\mathbb{R}$. Suppose that, for $\mathbb{P}_Y$-almost all $y$, $f_n(X,y) \xrightarrow{\mathbb{P}} f(X,y)$ when $n \to +\infty$. Then $f_n(X,Y) \xrightarrow{\mathbb{P}} f(X,Y)$ when $n \to +\infty$.*

Next, since the outer sums in (6.3) are both finite, we derive $B(\mathbf{a}^i, x, n') \xrightarrow{P} B(\mathbf{a}^i, x)$ when $n' \to +\infty$. Thus, when $a_0^i \neq 0$, for any subsequence $(n') \subset (n)$, there is some further subsequence $(n'') \subset (n')$ such that $B(\mathbf{a}^i, x, n'') \to B(\mathbf{a}^i, x)$ for $P$-almost all $x$.

If $a_0^i = 0$, the inner sum in (6.3) is empty and reduces to $f_i^n(t_1, \dots, t_{d_i})$. But taking eventually a subsequence, for almost all $t_1, \dots, t_{d_i}$, we have

$$f_i^n(t_1,\dots,t_{d_i}) \to f_i(t_1,\dots,t_{d_i}).$$

Since the outer sums in (6.3) are still both finite, we infer once more that for any subsequence $(n') \subset (n)$ there is some further $(n'') \subset (n')$ with $B(\mathbf{a}^i, x, n'') \to B(\mathbf{a}^i, x)$, $n'' \to +\infty$, for $P$-almost all $x$.

In both cases ($a_0^i$ is zero or not), the convergence in probability is proved. ∎

**6.2. Study of the coefficients $B(\mathbf{a}^i, y, n)$.** We deal now with $B(\mathbf{a}^i, y, n)$ for $y \in V(x)$ and to this end we adapt the study of the coefficients $B(\mathbf{a}^i, x, n)$ given in the proof of Lemma 6.2. First, we have to study the jumps and the jump times of $y \in V(x)$. Note that the (technical) choice of the parameters of local fields $l^1, \dots, l^p$ (see around (4.3)) are required specifically for this study. This preliminary work has already been done in [5], pp. 66–67, to which we will refer for a more precise justification.

LEMMA 6.4 (Jumps of $y \in V(x)$). *Let $y \in V(x)$, the neighbourhood of $x$ defined in* (4.6). *The list of the jump times of $x$ is denoted by $(t_i)_i$ and that of $y$ by $(s_i)_i$. We have*

$$T_\varepsilon(y) = \big(\rho^{-1}(t_1), \dots, \rho^{-1}(t_N)\big)$$

*for some increasing continuous bijection* $\rho$ *of* $[0,1]$ *realizing the Skorokhod distance between* $x$ *and* $y$. *Moreover*, *the jump times* $s_i$, $i>0$, *of* $y$ *satisfy:*

$$\omega_y^i(s_k)=0 \quad \text{if } s_k \notin \cup_{j=1}^{b_i}\Delta_j^i,$$
$$\omega_y^i(s_k)\neq 0 \quad \text{if } s_k \in \cup_{j=1}^{b_i}\Delta_j^i \text{ and } s_k=\rho^{-1}(t_j^i).$$

Proof. By the definition of Skorokhod's topology (see [2]), let $\rho\in\Lambda([0,1])$, the set of increasing continuous bijections of $[0,1]$, with

$$\sup_{t\in[0,1]}\big|x\big(\rho(t)\big)-y(t)\big|<\delta_2 \qquad \text{and} \qquad \sup_{t\in[0,1]}|\rho(t)-t|<\delta_2,$$

where $\delta_2$ is given in (4.3). We have

$$\delta_x\big(\rho(t)\big)-2\delta_2<\delta_y(t)<\delta_x\big(\rho(t)\big)+2\delta_2,$$
$$\big|\delta_x\big(\rho(t)\big)\big|-2\delta_2<|\delta_y(t)|<\big|\delta_x\big(\rho(t)\big)\big|+2\delta_2.$$

First $\rho^{-1}(t_i)\in\Delta_i=(t_i-\beta,t_i+\beta)$ because $|\rho(t_i)-t_i|<\delta_2$ and $\delta_2<\beta$. Moreover, we have

$$\big|\delta_y\big(\rho^{-1}(t_i)\big)\big|>|\delta_x(t_i)|-2\delta_2\geqslant 2\varepsilon_0-\frac{1}{2}\varepsilon_0=\frac{3}{2}\varepsilon_0>\varepsilon_0>\varepsilon_i.$$

If $t\in\Delta_i\backslash\{\rho^{-1}(t_i)\}$, we also have $\rho(t)\in\Delta_i'=(t_i-\delta_1,t_i+\delta_1)$ because it follows that $|\rho(t)-t|<\delta_2$ and $\beta=\delta_1-\delta_2$. Consequently, since

- $\big|\delta_x\big(\rho(t)\big)\big|\leqslant\varepsilon_0/2$ because $\rho(t)\neq t_i$ and $t_i$ is the unique time in $\Delta_i'$ when there occurs a jump of $x$ larger than $\varepsilon_0/2$,
- $2\delta_2<\varepsilon_1-\varepsilon_0/2$ by the choice of $\delta_2$ in (4.3),

we have

$$|\delta_y(t)|<\big|\delta_x\big(\rho(t)\big)\big|+2\delta_2\leqslant\frac{\varepsilon_0}{2}+2\delta_2<\varepsilon_1\leqslant\varepsilon_i.$$

For $t\in\Delta_i$:

if $t=\rho^{-1}(t_i)$, then $t\in\Delta_i$, $|\delta_y(t)|>\varepsilon_i$, $\delta_y(t)$ has the same sign as $\delta_x(t_i)$;

if $t\neq\rho^{-1}(t_i)$, then $|\delta_y(t)|<\varepsilon_i$.

Observe moreover that for $t\in U_i^\varepsilon$, $t\neq\rho^{-1}(t_i)$:

$$|\delta_y(t)|\leqslant\big|\delta_x\big(\rho^{-1}(t)\big)\big|+2\delta_2<|\delta_x(t_i')|+2\delta_2 \tag{6.5}$$

because $\rho^{-1}(t)\neq t_i$ implies $\big|\delta_x\big(\rho^{-1}(t)\big)\big|<|\delta_x(t_i')|$ and

$$\big|\delta_y\big(\rho^{-1}(t_i)\big)\big|>|\delta_x(t_i)|-2\delta_2. \tag{6.6}$$

From (4.3) we get $\delta_2<\frac{1}{4}\min_{i=1,\dots,N}\{|\delta_x(t_i)|-|\delta_x(t_i')|\}$, and from (6.5) and (6.6) we deduce that $|\delta_y(t)|<\big|\delta_y\big(\rho^{-1}(t_i)\big)\big|$. Then we have $\rho^{-1}(t_i)=T_{U_i^\varepsilon}(y)$ and

$$\big(\rho^{-1}(t_1),\dots,\ \rho^{-1}(t_N)\big)=T_\varepsilon(y).$$

This argument justifies also the second part of Lemma 6.4. ∎

LEMMA 6.5. *In $V(x)$, we have $B(\mathbf{a}^i, y, n) \xrightarrow{P} B(\mathbf{a}^i, y)$ when $n \to +\infty$.*

P r o o f. We study the coefficients $B(\mathbf{a}^i, y, n)$ just like we did for $B(\mathbf{a}^i, x, n)$ but using additionally the examination of the jump times of $y$ in Lemma 6.4. Here, from Lemma 6.4 we get $\prod_{j\in I_1} w_y^1(s_j) = \pm\tau^{a_1^i}$ for $A_{b_1}^{a_1^i}$ choices of $s_j$, $j \in I_1$, else it is 0. Doing the same for the other products, we have

$$\Big(\prod_{j\in I_1} w_x^1(s_j)\Big)\dots\Big(\prod_{j\in I_p} w_x^p(s_j)\Big) = \pm\tau^{a_1^i+\dots+a_p^i}$$

for $A(\mathbf{a}^i) := A_{b_1}^{a_1^i} \times \dots \times A_{b_p}^{a_p^i}$ choices of index $j$, else the product is zero. Using the symmetry of the kernels $f_i$ and the nullity of $f_i$ on the diagonals, we can thus rewrite

(6.7)
$$B(\mathbf{a}^i, y) = \sum_{\substack{\{I_k\}\text{ partition of}\\ \{1,\dots,d_i\},\ \mathrm{card}\, I_k = a_k^i}}^{(1)} \ \sum_{\substack{A(\mathbf{a}^i)\text{ choices of}\\ s_{a_0^i+1},\dots,s_{d_i}}}^{(3)} \pm\tau^{a_1^i+\dots+a_p^i} \sum_{s_1,\dots,s_{a_0^i}} \Big(\prod_{j\in I_0}\delta_x(s_j)\Big) f_i(s_1,\dots,s_{d_i}).$$

Observe again that the outer sums in (6.7), i.e. $\sum^{(1)}$ and $\sum^{(3)}$, are both finite. The same computations hold true for the coefficients $B(\mathbf{a}^i, y, n)$ with $f_i^n$ in place of $f_i$.

In order to study the convergence of the coefficients $B(\mathbf{a}^i, y, n)$, we first deal with the convergence of the inner sum

$$\sum_{s_1,\dots,s_{a_0^i}} \Big(\prod_{j\in I_0}\delta_y(s_j)\Big) f_i^n(s_1,\dots,s_{d_i}) \quad \text{as } n \to +\infty,$$

where $s_{a_0^i+1},\dots,s_{d_i}$ in $f_i^n$ appear as parameters.

As in the proof of Lemma 6.2 for the case with $x$, when $a_0^i \neq 0$, this sum can be seen from (3.2) as an MSI and since from $f_i^n \to f_i$ in $L^\alpha(\log_+)^{d_i-1}([0,1]^{d_i})$, eventually taking some subsequence, we derive the convergence

$$f_i^n(\cdot, s_{a_0^i+1},\dots,s_{d_i}) \to f(\cdot, s_{a_0^i+1},\dots,s_{d_i})$$

in $L^\alpha(\log_+)^{d_i-1}([0,1]^{a_0^i})$, and thus also in $L^\alpha(\log_+)^{a_0^i-1}([0,1]^{a_0^i})$ for almost all $s_{a_0^1+1},\dots,s_{d_i}$. Thus, from Proposition 3.1, when $a_0^i \neq 0$, we have

$$I_{a_0^i}\big(f_i^n(\cdot, t_{a_0^i+1},\dots,t_{d_i})\big) \xrightarrow{\mathbb{P}} I_{a_0^i}\big(f(\cdot, t_{a_0^i+1},\dots,t_{d_i})\big).$$

Arguing as in Proposition 3.2, we rewrite when $n \to +\infty$

$$\sum_{t_1,\dots,t_{a_0^i}} \Big(\prod_{j\in I_0}\delta_x(t_j)\Big) f_i^n(t_1,\dots,t_{d_i}) \xrightarrow{P} \sum_{t_1,\dots,t_{a_0^i}} \Big(\prod_{j\in I_0}\delta_y(t_j)\Big) f_i(t_1,\dots,t_{d_i})$$

for almost all $s_{a_0^1+1}, \dots, s_{d_i}$. Applying first Lemma 6.3 and using next the finiteness of the outer sums in (6.7), we have $B(\mathbf{a}^i, y, n) \xrightarrow{P} B(\mathbf{a}^i, y)$ as $n \to +\infty$, and thus for any subsequence $(n') \subset (n)$, there is some further subsequence $(n'') \subset (n')$ such that $B(\mathbf{a}^i, y, n'') \to B(\mathbf{a}^i, y)$ for $P$-almost all $y \in V(x)$ in the case where $a_0^i \neq 0$.

If $a_0^i = 0$, as in the case for $x$ in the proof of Lemma 6.4, the inner sum in (6.7) is empty and reduces to $f_i^n(s_1, \dots, s_{d_i})$. But taking eventually a subsequence, for almost all $s_1, \dots, s_{d_i}$, we have $f_i^n(s_1, \dots, s_{d_i}) \to f_i(s_1, \dots, s_{d_i})$. Since the outer sums in (6.3) are still both finite, we derive once more that for any subsequence $(n') \subset (n)$ there is some further $(n'') \subset (n')$ with $B(\mathbf{a}^i, y, n'') \to B(\mathbf{a}^i, y)$ for $P$-almost all $y$.

We thus have for $P$-almost all $y \in V(x)$ the convergence of the coefficients $B(\mathbf{a}^i, y, n'')$ of $\varphi^i_{n'',y}$ to the coefficient $B(\mathbf{a}^i, y)$ of $\varphi^i_y$, and the convergence in probability follows. ∎

Finally, we conclude now this section with the proof of (5.9). From the expression in (6.1) and from Lemma 6.5 we derive for any subsequence $(n') \subset (n)$ that there is some further subsequence $(n'') \subset (n')$ such that for $P$-almost all $y \in V(x)$ we have the convergence of $\varphi_{n'',y}$ to $\varphi_y$ in the local Sobolev space $W^{p,1}_{\text{loc}}(\mathbb{R}^p, \mathbb{R}^p)$. Moreover, from Lemma 6.1, under the hypothesis (**H**), for $P$-almost all $y \in V(x)$ we have $J_y(\mathbf{c}) \neq 0$ for almost all $\mathbf{c}$. We can thus apply Proposition 2.1 (Corollary 4 in [1]) to derive the convergence (5.9) when $n'' \to +\infty$, that is

$$\lambda_{[0,\epsilon]^p} \varphi^{-1}_{n'',y} \xrightarrow{\text{var}} \lambda_{[0,\epsilon]^p} \varphi^{-1}_y .$$

## 7. CONCLUSION

For $P$-almost all $y \in V(x)$, the convergence in (5.9) has been derived for some subsequence $(n'')$ taken from any subsequence $(n') \subset (n)$. Finally, returning to the second term on the right-hand side of (5.8), we derive for all $\epsilon > 0$ and for a further subsequence $(n'') \subset (n')$

$$\lim_{n'' \to +\infty} \|Q_\epsilon F_\epsilon^{-1} - Q_\epsilon (F_\epsilon^{n''})^{-1}\| = 0.$$

From (5.8) we thus have

$$\overline{\lim_{n'' \to +\infty}} \|P^x F^{-1} - P^x (F^{n''})^{-1}\| \leqslant 3\epsilon.$$

Since $\epsilon > 0$ is arbitrary, we obtain

$$P^x (F^n)^{-1} \xrightarrow{\text{var}} P^x F^{-1} \quad \text{as } n \to +\infty.$$

Finally, gathering together all the steps, first we have

$$P^x(F^n)^{-1} \xrightarrow{\text{var}} P^x F^{-1} \quad \text{as } n \to +\infty,$$

and next, by localization,

$$P_{\mathbb{D}(\varepsilon)}(F^n)^{-1} \xrightarrow{\text{var}} P_{\mathbb{D}(\varepsilon)} F^{-1} \quad \text{as } n \to +\infty.$$

Finally, by approximation, we get

$$P(F^n)^{-1} \xrightarrow{\text{var}} PF^{-1} \quad \text{as } n \to +\infty.$$

Thus, we have proved the convergence in variation of $\mathcal{L}\big(S_{d_1}(f_1^n), \ldots, S_{d_p}(f_p^n)\big)$ to $\mathcal{L}\big(S_{d_1}(f_1), \ldots, S_{d_p}(f_p)\big)$. By the representation theorem the same holds true for the law of $\big(I_{d_1}(f_1^n), \ldots, I_{d_p}(f_p^n)\big)$ to those of $\big(I_{d_1}(f_1), \ldots, I_{d_p}(f_p)\big)$. This completes the proof of Theorem 2.1.

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*Received on 5.10.2006;
revised version on 14.3.2007*